Curvature and the Euler Characteristic in Geometry and Algebra

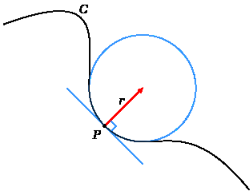
Carlos Florentino[1,2]

[1]Departmento de Matemática and [2]CMAFcIO
Universidade de Lisboa

Ciências Research Day 26/10/2022

Outline

Gaussian Curvature, I



Take a point P of a curve on a plane. **Curvature** is the **inverse** of the **radius** of the **oscullating circle**:

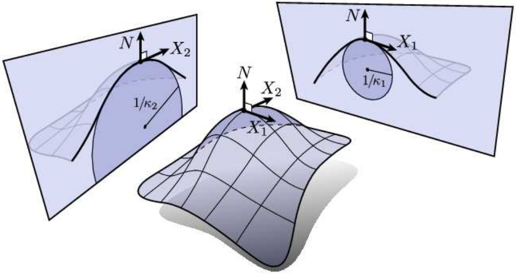
$$\kappa = \frac{1}{r}$$

Smaller oscullating circles have greater curvature, and vice-versa. At a portion of straight line the **curvature is zero**.

What about surfaces?

Gaussian Curvature, II - *Theorema Egregium*

C. F. Gauss - 1777-1855



$$K := \pm\kappa_1\kappa_2, \quad \kappa_1 = \min\kappa, \quad \kappa_2 = \max\kappa$$

Gauss' Theorema Egregium

Scalar (Gaussian) curvature K only depends on the **metric**. In particular, it is invariant under **bending**!

However: K is not invariant under **streching**!

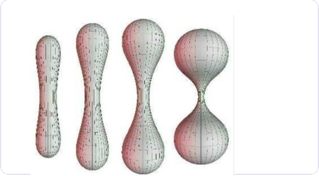
Total curvature and Gauss-Bonnet

Definition: The ***total curvature*** of a closed surface (with no boundary) is

$$TK_S := \int_S K$$

Theorem (Gauss-Bonnet)

Total curvature is invariant under **all deformations !!** (except cutting/glueing)



One then computes:

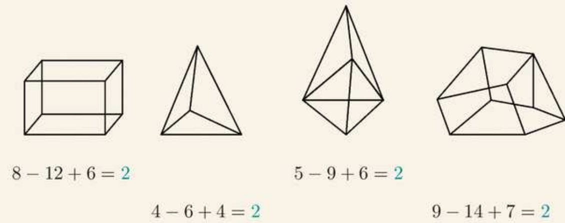
S				$\cdots$	g-holed torus
TK_S	4π	0	-4π	$\cdots$	$2\pi(2-2g) = 2\pi\chi_S$

Euler characteristic of S: $\chi_S = 2 - 2g$

The Euler characteristic χ

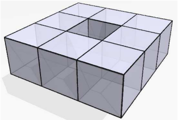
L. Euler, 1707-83

For all convex polyhedra:
$$V - E + F = 2 \quad !!$$



Therefore, for any *"triagulated" surface*:

$$TK_S = \int_S K = 2\pi(V - E + F).$$



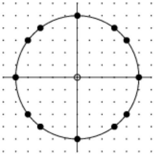
$$V - E + F = 32 - 64 + 32 = 0$$

Solving Polynomial equations in integers

Many solutions: $(3,4,5)$, $(5,12,13)$, etc, are solutions to the equation: $x^2+y^2=z^2$, called Pitagorian triples.

No solutions: (Fermat's "Last theorem" / Wiles-Taylor, 1994): There are no integer solutions to

$$x^n+y^n+z^n=0, \qquad xyz\neq 0, \qquad n\geq 3$$



Theorem (C. G. J. Jacobi 1829)

The number of integer solutions to
$x^2+y^2=n$ is $4d_1(n)-4d_3(n)$!!

Theorem (G. Faltings 1983)

If $p(x,y,z)=0$, a polynomial with integer coefficients, describes a smooth surface S with $\chi_S<0 \quad \Leftrightarrow \quad g\geq 2$, then it has a **finite number** of (primitive) integer solutions!

Algebraic varieties and Lie groups

Simplified Definition: An *algebraic variety* is the set of solutions of a system of polynomials equations in several variables.

Sophus Lie, 1842-99

Simplified Definition: A **Lie group** is a closed subgroup of the group of invertible matrices ($\det \neq 0$).

$$GL(n,\mathbb{C}) = \{A \in Mat_{n\times n}(\mathbb{C}) \ : \ \det A \neq 0\}$$
$$SO(n) = \{A \in Mat_{n\times n}(\mathbb{R}) \ : \ A^t A = I, \quad \det A = 1\}$$

The *nicest Theorem* that does not have a name:

Every compact Lie group of positive dimension has zero Euler characteristic.

Commuting group-matrices

Two square matrices A and B commute if $AB = BA$.

Let G be a Lie group, $n \in \mathbb{N}$. Define:

$$C_n G = \{\ (A_1, A_2, \cdots, A_n) \in G^n\ :\ A_i A_j = A_j A_i\ \}^\circ$$

Why study commuting group-matrices?

	Classical Physics	Quantum Physics
Observable	Function f	Operator A
State	point x in phase space	element ψ in Hilbert space
Measurement	evaluation of $f(x)$	$\langle \psi \| A \| \psi \rangle$

Operators that commute *behave classically*:
associated observables can be simultaneously measured.

How many commuting group-matrices are there over $\mathbb{F}_p$?

Let p be a prime number, and $\mathbb{F}_p$ be the field with p elements: $\mathbb{F}_p = \{0, 1, \cdots, p-1\}$. Recall that $C_n G$ is given by polynomial equations.

Theorem (F-Lawton-Silva, '21)

The number of points in $\mathbb{F}_p$ of $C_n G/G$ is:

$$\frac{1}{|W|}\sum_{g\in W}\det(pI-g)^n.$$

The Euler characteristic is $\frac{1}{|W|}\sum_{g\in W}\det(I-g)^n$.

Example

3 commuting matrices in $SO(5,\mathbb{F}_p)=\{A\in Mat_{5\times 5}(\mathbb{F}_p) : A^tA=I_5\}$, up to conjugation, have exactly

$$p^6+3p^4+6p^2$$

points in $\mathbb{F}_p$.

Conclusions and Outlook

- The Euler characteristic is one of the simplest properties of a space
- It is also extremely useful – Related to total curvature, counting, and many other global properties.
- There are also refined invariants: Poincaré polynomial, Hodge-Deligne/Serre polynomials, etc, these describe how the space is decomposed into subspaces of smaller dimensions.
- Qualitatively different spaces cannot have the same invariants!!
- Very useful for classification:
    - Vector spaces – dimension
    - Surfaces – genus, etc
    - Varieties, Lie Groups, ...

Intriguing Question: Can one detect curvature by counting (integer) solutions of polynomial equations (or vice-versa)?

For Further Reading

Obrigado pela atenção!

D. Richeson, *Euler's Gem: The Polyhedron Formula and the Birth of Topology*, Princeton University Press, 2012.

G. Faltings, *Finiteness theorems for abelian varieties over number fields*, Inventiones Mathematicae (in German) **73** (1983) 349–366.

C. Florentino, S. Lawton and J. Silva, *Mixed Hodge structures on character varieties of nilpotent groups*, arXiv:2110.07060 [math:AG].

C. Florentino and J. Silva, *Hodge-Deligne polynomials of character varieties of abelian groups*, Open Math. **19** (2021) 338-362.